ROTATIONAL FREEDOM

Exploring the limits to which functions may be rotated while remaining a function

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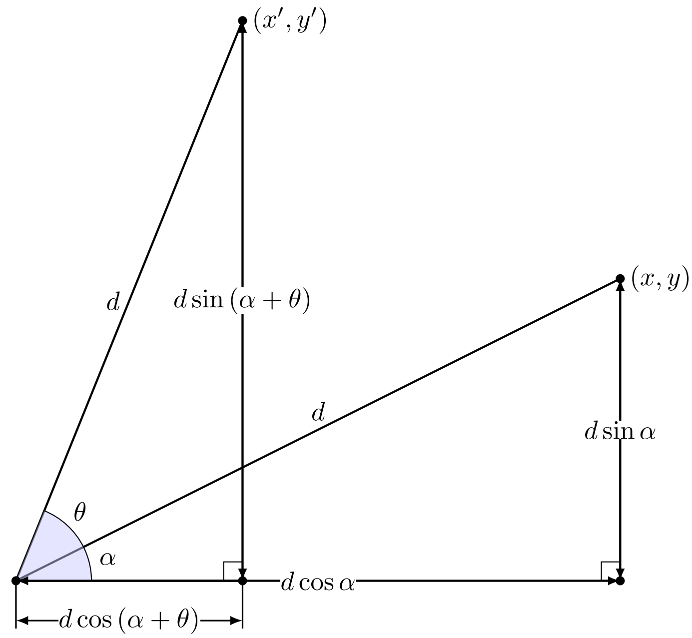
1 Rotating a function about the origin

1.1 Rotating a point

In order to rotate a function about the origin, one must first use elementary geometry to rotate singular points. Let (x, y) be the point we wish to rotate an angle of θ counter-clockwise about the origin to a new point (x', y'). We may assign d to be the distance between the point and the origin, at an angle of α to the positive x-axis before rotation and an angle of $\alpha + \theta$ after rotation.



Using the addition formulae for sin and cos we can expand both $d \sin(\alpha + \theta)$ and $d \cos(\alpha + \theta)$ in order to get an expression for both x' and y'.

$$\begin{aligned} d \sin(\alpha + \theta) &= d \cos \alpha \sin \theta + d \sin \alpha \cos \theta \\ &= x \sin \theta + y \cos \theta \end{aligned} \tag{1}$$

$$\begin{aligned} d \cos(\alpha + \theta) &= d \cos \alpha \cos \theta - d \sin \alpha \sin \theta \\ &= x \cos \theta - y \sin \theta \end{aligned} \tag{2}$$

From (1) and (2), one can then see that $(x', y') = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$.

1.2 Rotating a curve

As we now know how to rotate a singular point about the origin, we may extend to a curve. A curve is simply a collection of points and so we may think of a method that applies our previous technique for singular points to each point in this collection. We can do this using parametric equations. For a curve, $y = f(x)$, one can map both the domain and range to two separate parametric equations.

$$y = t\sin\theta + f(t)\cos\theta \tag{3}$$

$$x = t\cos\theta - f(t)\sin\theta \tag{4}$$

We can assume $f(t)$ to be polynomial for now, and let $t \in \mathbb{R}$. Now, if we plot this parametric curve using our two equations, changing θ would rotate each point on the curve, and thereby rotate the whole curve.

1.3 Example of rotation

We can visualise an example using $y = \sin x$. This can be done using any graphing software such as Desmos.

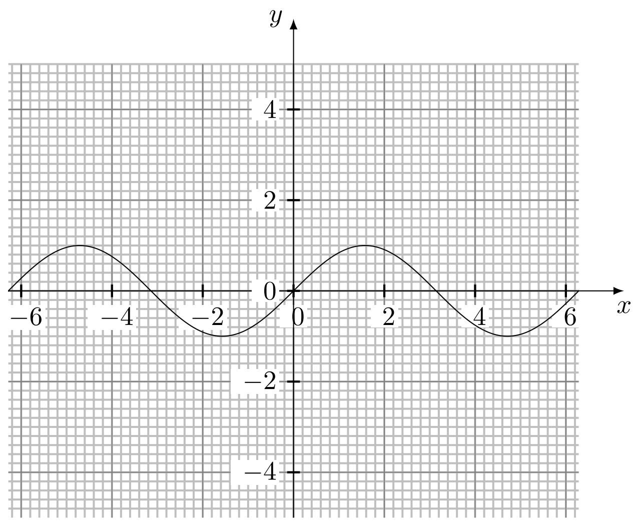


Figure 1: $\theta = 0$

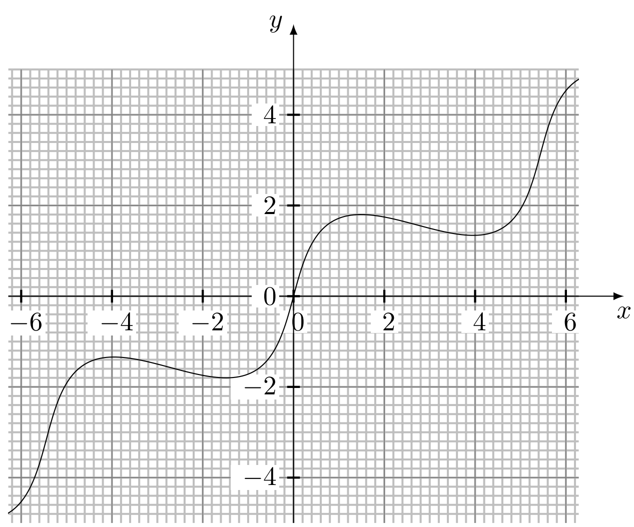


Figure 2: $\theta = \frac{\pi}{6}$

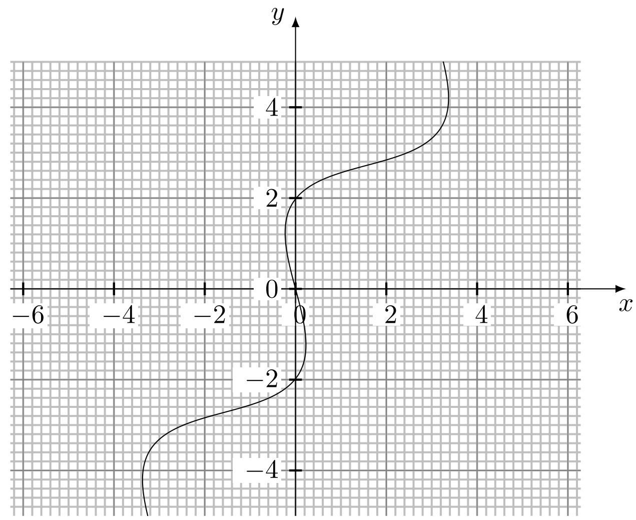


Figure 3: $\theta = \frac{\pi}{3}$

Note the difference between Figure 2 and Figure 3, although you cannot verify, you can see that Figure 2 remains a true function, while Figure 3 does not. One can test for a function via the vertical line test. If you can draw a vertical line that intersects the curve more than one time, it is not a true function.

2 Maintaining realness

2.1 Requirements for realness

A function must have a single output for every input. The relations which are functions are one-to-one and many-to-one. Other relations are displayed below.

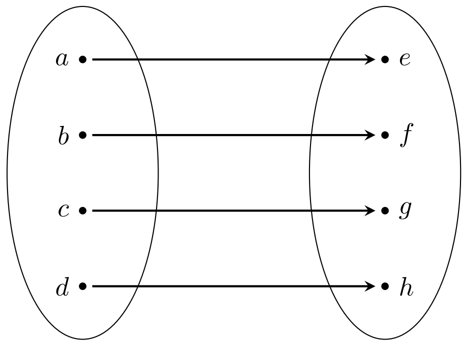


Figure 4: One-to-one

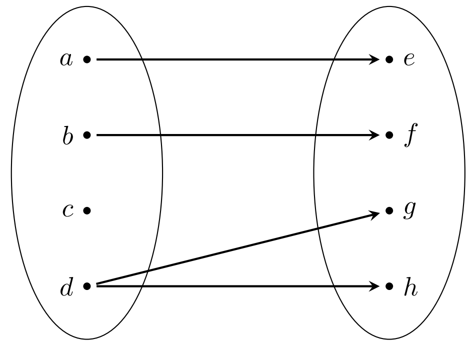


Figure 6: One-to-many

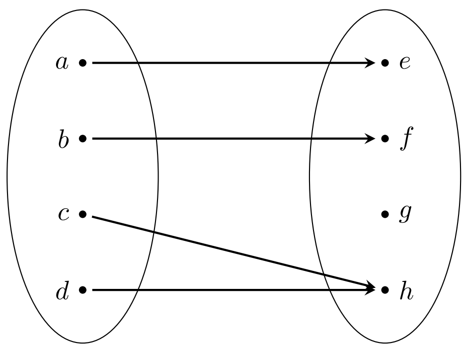


Figure 5: Many-to-one

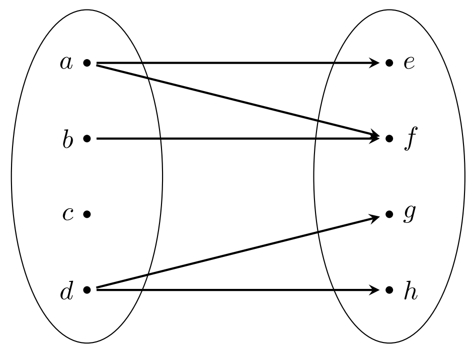


Figure 7: Many-to-many

In order for the function to remain real, it must be either one-to-one or many-to-one, this means that the parametric equation representing our domain, $x = t\cos\theta - f(t)\sin\theta$, must be a one-to-one function. This implies that the function must be either an increasing function or a decreasing function. However, we may encapsulate both of these statements by requiring the function to have no stationary points.

2.2 Satsifying those requirements

In order for the function to have no stationary points, the derivative function must have no roots.

$$x = t\cos\theta - f(t)\sin\theta$$

$$\frac{dx}{dt} \neq 0$$

$$\implies \cos\theta - f'(t)\sin\theta \neq 0$$

$$\implies \cos\theta \neq f'(t)\sin\theta$$

$$\implies \cot\theta \neq f'(t) \tag{5}$$

For all $t \in \mathbb{R}$, (5) must be satisfied in order for the function to remain real after being rotated θ about the origin.

2.3 Proof: No function has full real revolutionary freedom

For a function to have a full range of revolutionary freedom, it must be rotatable for all $0 \leq \theta < 2\pi$. The function $\cot\theta$ ranges over all real numbers for this domain, see Figure 8. This means that our condition established in (5) cannot be satisfied for any $f(t)$ as there will always be an intersection between $f'(t)$ and $\cot\theta$.

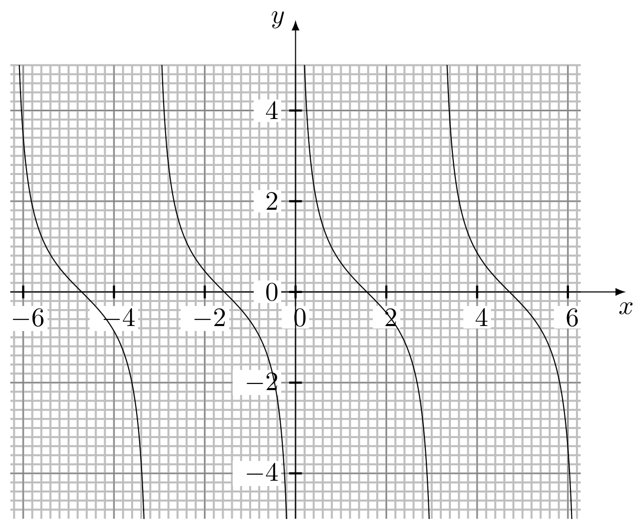


Figure 8: $y = \cot\theta$

This can be extended to rotation around any point (a,b). If we rotate $f(x)$ from (a, b), when looking at values of θ it can be rotated while being real, it is the same as looking at the rotation of $f(x+a) - b$ from the origin. Let $g(x) = f(x+a) - b \implies g'(x) = f'(x+a)$

The range of $f'(x+a)$ is equal to the range of $f'(x)$ Therefore, rotating from (a,b) preserves the values of θ for which a function can be rotated while staying real. Therefore, no function rotated from any point can be rotated fully while staying real.

2.4 Finding the revolutionary freedom of a function

In order to find the revolutionary freedom of a function, we must find all values of θ that satisfy the condition $\cot\theta \neq f'(t)$ for all $t \in \mathbb{R}$.
Let a and b bound $f'(t)$ so that $a \leq f'(t) \leq b$.
Therefore, the values of theta which are invalid for rotation are in $\{\theta : a \leq \cot\theta \leq b\}$.
We can then take the complement of this set, then intersect it with $\{\theta : 0 \leq \theta < 2\pi\}$.
This intersection is the set of values of θ by which $f(x)$ can be rotated.

We can try this method with $f(x) = sin(x)$. The derivative function $f'(x) = cos(x)$ is bounded by -1 and 1.

$$\{\theta : -1 \leq \cot\theta \leq 1\}$$

Intersecting the complement of this set with $\{\theta : 0 \leq \theta < 2\pi\}$, we have

$$\left\{\theta : 0 \leq \theta < \frac{\pi}{4}, \frac{3\pi}{4} < \theta < \frac{5\pi}{4}, \frac{7\pi}{4} < \theta \leq 2\pi\right\}$$

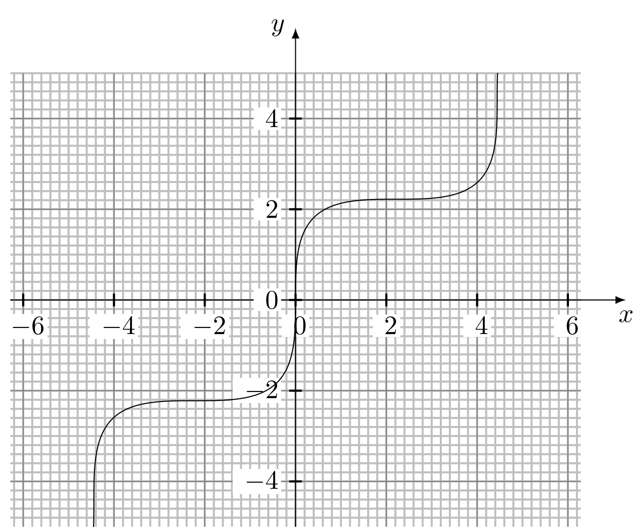


Figure 9: $\theta = \frac{\pi}{4}$

3 Going Further

Throughout this paper, we have explored the limits to which functions may be rotated while remaining well-defined. This journey is still far from over, however. We plan on extending this paper with further ideas including: the analyses of different families of functions and exploring how the integral of the function changes as it rotates. We shall also be looking at different applications of rotational freedom, including a study on how rotational freedom can be used in the agriculture industry for yield optimisations. We hope you have enjoyed reading this paper and have learned something new or interesting, and encourage you to go further and extend on this new knowledge.